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On the asymptotic behavior of nonoscillatory solutions of second order quasilinear ordinary differential equations

Manabu Naito

Department of Mathematics, Faculty of Science, Ehime University, Matsuyama 790-8577, Japan

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ABSTRACT

In this paper second order quasilinear ordinary differential equations are considered, and a necessary and sufficient condition for the existence of a slowly growing positive solution is established. Moreover, the precise asymptotic forms as $t \to \infty$ of slowly growing positive solutions and slowly decaying positive solutions are obtained.

1. Introduction

In this paper we are concerned with the second order quasilinear ordinary differential equation

$$\left(p(t)\left|x'\right|^{\alpha} \operatorname{sgn} x'\right)' + q(t)|x|^{\beta} \operatorname{sgn} x = 0, \tag{1.1}$$

where the following conditions are always assumed:

$$\alpha \text{ and } \beta \text{ are positive constants;} \tag{1.2}$$

$$p \in C[a,\infty) \quad \text{and} \quad p(t) > 0 \quad \text{on } [a,\infty); \tag{1.3}$$

$$q \in C[a,\infty) \quad \text{and} \quad q(t) \geqslant 0 \quad \text{on } [a,\infty), \quad \text{and} \quad q(t) \not\equiv 0 \quad \text{on } [b,\infty) \text{ for any } b \geqslant a. \tag{1.4}$$

By a solution of (1.1) we mean a real-valued function $x = x(t)$ such that $x \in C^1[T,\infty)$, $T \geqslant a$, and $p|x'|^{\alpha} \operatorname{sgn} x' \in C^1[T,\infty)$ and $x(t)$ satisfies (1.1) at every point of $[T,\infty)$, where T may depend on $x(t)$. A solution $x(t)$ of (1.1) is said to be *oscillatory* if there is a sequence $\{t_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} t_i = \infty$ and $x(t_i) = 0$ $(i = 1, 2, \ldots)$. If a solution $x(t)$ of (1.1) is not oscillatory, then it is said to be *nonoscillatory*. In other words, a solution $x(t)$ of (1.1) is called nonoscillatory if $x(t)$ is eventually positive or eventually negative. If $x(t)$ is a solution of (1.1), then so is $-x(t)$. Therefore there is no loss of generality in assuming that a nonoscillatory solution of (1.1) is eventually positive.

Recently there has been much interest on the study of oscillatory and nonoscillatory behavior of solutions of (1.1). It is known that the case

$$\int_a^{\infty} \frac{1}{p(t)^{1/\alpha}}\, dt = \infty \tag{1.5}$$

E-mail address: mnaito@math.sci.ehime-u.ac.jp.

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and the case

$$\int_a^\infty \frac{1}{p(t)^{1/\alpha}}\,dt < \infty \tag{1.6}$$

give a different aspect on the existence and asymptotic behavior of nonoscillatory solutions of (1.1).

First, consider the case where (1.5) holds. In this case, we define $P(t)$ by

$$P(t) = \int_a^t \frac{1}{p(s)^{1/\alpha}}\,ds, \quad t \geqslant a. \tag{1.7}$$

It is easily seen that (Elbert [1], Elbert and Kusano [2]) if $x(t)$ is an eventually positive solution of (1.1), then there are positive constants c_1 and c_2 such that $c_1 \leqslant x(t) \leqslant c_2 P(t)$ for all large t, more precisely, exactly one of the following three conditions is satisfied:

$$\lim_{t\to\infty} x(t) \quad \text{exists and is a positive finite number;} \tag{1.8}$$

$$\lim_{t\to\infty} x(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{x(t)}{P(t)} = 0; \tag{1.9}$$

$$\lim_{t\to\infty} \frac{x(t)}{P(t)} \quad \text{exists and is a positive finite number.} \tag{1.10}$$

Further we have the following results [1,2,7,8].

(A) Eq. (1.1) has an eventually positive solution $x(t)$ satisfying (1.8) if and only if

$$\int_a^\infty \left[\frac{1}{p(t)} \int_t^\infty q(s)\,ds\right]^{1/\alpha} dt < \infty. \tag{1.11}$$

(B) Eq. (1.1) has an eventually positive solution $x(t)$ satisfying (1.10) if and only if

$$\int_a^\infty q(t)P(t)^\beta\,dt < \infty. \tag{1.12}$$

(C) Let $\alpha < \beta$. Eq. (1.1) has an eventually positive solution if and only if (1.11) is satisfied.

(D) Let $\alpha > \beta$. Eq. (1.1) has an eventually positive solution if and only if (1.12) is satisfied.

In this paper we refer to eventually positive solutions $x(t)$ satisfying (1.9) as *slowly growing positive solutions*.

We next consider the case where (1.6) holds. In this case, define $\pi(t)$ by

$$\pi(t) = \int_t^\infty \frac{1}{p(s)^{1/\alpha}}\,ds, \quad t \geqslant a. \tag{1.13}$$

If $x(t)$ is an eventually positive solution of (1.1), then there are positive constants c_1 and c_2 such that $c_1\pi(t) \leqslant x(t) \leqslant c_2$ for all large t, and exactly one of the following three conditions is satisfied (Kusano, Ogata and Usami [5]):

$$\lim_{t\to\infty} \frac{x(t)}{\pi(t)} \quad \text{exists and is a positive finite number;} \tag{1.14}$$

$$\lim_{t\to\infty} \frac{x(t)}{\pi(t)} = \infty \quad \text{and} \quad \lim_{t\to\infty} x(t) = 0; \tag{1.15}$$

$$\lim_{t\to\infty} x(t) \quad \text{exists and is a positive finite number.} \tag{1.16}$$

Moreover it is known that the following results hold [5].

(a) Eq. (1.1) has an eventually positive solution $x(t)$ satisfying (1.14) if and only if

$$\int_a^\infty q(t)\pi(t)^\beta\,dt < \infty. \tag{1.17}$$

(b) Eq. (1.1) has an eventually positive solution $x(t)$ satisfying (1.16) if and only if

$$\int_a^\infty \left[\frac{1}{p(t)} \int_a^t q(s)\,ds\right]^{1/\alpha} dt < \infty. \tag{1.18}$$

(c) Let $\alpha < \beta$. Eq. (1.1) has an eventually positive solution if and only if (1.17) holds.
(d) Let $\alpha > \beta$. Eq. (1.1) has an eventually positive solution if and only if (1.18) holds.

In this paper we refer to eventually positive solutions $x(t)$ satisfying (1.15) as *slowly decaying positive solutions*.

It is worth noting that there is a remarkable "duality" between the cases of (1.5) and (1.6). Very recently, the following theorem has been proved by Kamo and Usami [4].

Theorem 1.1. *Suppose that* (1.6) *holds. Let* $\alpha > \beta$. *Eq.* (1.1) *has a slowly decaying positive solution if and only if*

$$\int_a^\infty \left[\frac{1}{p(t)}\int_a^t q(s)\,ds\right]^{1/\alpha} dt < \infty \quad \textit{and} \quad \int_a^\infty q(t)\pi(t)^\beta\,dt = \infty. \tag{1.19}$$

By the "duality" between (1.5) and (1.6), it is natural to conjecture the following theorem.

Theorem 1.2. *Suppose that* (1.5) *holds. Let* $\alpha > \beta$. *Eq.* (1.1) *has a slowly growing positive solution if and only if*

$$\int_a^\infty q(t)P(t)^\beta\,dt < \infty \quad \textit{and} \quad \int_a^\infty \left[\frac{1}{p(t)}\int_t^\infty q(s)\,ds\right]^{1/\alpha} dt = \infty. \tag{1.20}$$

In fact, Theorem 1.2 is partially proved. For the case $\alpha = 1 > \beta$, we can easily find that Theorem 1.2 is true (Kusano and Naito [6]). For the general case $\alpha > \beta$, the "if" part of Theorem 1.2 is also true (Elbert and Kusano [2]). In this paper we will give a proof of the "only if" part, and consequently we find that Theorem 1.2 is totally true.

Let $\alpha > \beta$. Then, under the condition that $p(t)$ and $q(t)$ behave like power functions, Kamo and Usami [3,4] have obtained precise asymptotic forms as $t \to \infty$ of slowly growing or slowly decaying positive solutions $x(t)$ of (1.1). Their results can be extended as follows.

Consider Eq. (1.1) together with the auxiliary equation

$$\left(p_0(t)\left|x'\right|^\alpha \operatorname{sgn} x'\right)' + q_0(t)|x|^\beta \operatorname{sgn} x = 0, \tag{1.21}$$

where α, β are positive constants, and $p_0, q_0 \in C[a,\infty)$, $p_0(t) > 0$, $q_0(t) > 0$ for $t \geqslant a$. We assume

$$p(t) \sim p_0(t) \quad \text{and} \quad q(t) \sim q_0(t) \quad \text{as } t \to \infty. \tag{1.22}$$

Here and hereafter, the notation $f(t) \sim g(t)$ as $t \to \infty$ means that

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1.$$

First, let us consider the case where (1.5) holds. By (1.22) we have

$$\int_a^\infty \frac{1}{p_0(t)^{1/\alpha}}\,dt = \infty.$$

Our attention is paid to a relationship with slowly growing positive solutions $x = x(t)$ of (1.1) and slowly growing positive solutions $x = x_0(t)$ of (1.21). Suppose that

$$\begin{cases} \text{there is a constant } C > 0 \quad \text{such that} \\ \displaystyle\int_a^t \left[\frac{1}{p(s)P(s)^\beta}\int_s^\infty q(r)P(r)^\beta\,dr\right]^{1/\alpha} ds \leqslant C\int_a^t \left[\frac{1}{p(s)}\int_s^\infty q(r)\,dr\right]^{1/\alpha} ds \quad \text{for all large } t. \end{cases} \tag{1.23}$$

Then we can conclude that

$$x(t) \sim x_0(t) \quad \text{as } t \to \infty. \tag{1.24}$$

That is to say, we have the following theorem.

Theorem 1.3. *In Eqs.* (1.1) *and* (1.21), *let* $\alpha > \beta$ *and suppose that* (1.5), (1.22) *and* (1.23) *hold. Let* $x(t)$ *and* $x_0(t)$ *be slowly growing positive solutions of* (1.1) *and* (1.21), *respectively. Then we have* (1.24).

Next consider the case where (1.6) holds. By (1.22) we have

$$\int_a^\infty \frac{1}{p_0(t)^{1/\alpha}}\,dt < \infty.$$

Thus our attention is paid to slowly decaying positive solutions of (1.1) and (1.21). The next conditions are used:

$$\int_a^{\infty}\left[\frac{1}{p(s)\pi(s)^{\beta}}\int_a^{s} q(r)\pi(r)^{\beta}\,dr\right]^{1/\alpha} ds < \infty \tag{1.25}$$

and

$$\begin{cases} \text{there is a constant } C > 0 \quad \text{such that} \\ \displaystyle\int_t^{\infty}\left[\frac{1}{p(s)\pi(s)^{\beta}}\int_a^{s} q(r)\pi(r)^{\beta}\,dr\right]^{1/\alpha} ds \leqslant C\int_t^{\infty}\left[\frac{1}{p(s)}\int_a^{s} q(r)\,dr\right]^{1/\alpha} ds \quad \text{for all large } t. \end{cases} \tag{1.26}$$

Theorem 1.4. *In Eqs.* (1.1) *and* (1.21), *let* $\alpha > \beta$ *and suppose that* (1.6), (1.22), (1.25) *and* (1.26) *hold. Let* $x(t)$ *and* $x_0(t)$ *be slowly decaying positive solutions of* (1.1) *and* (1.21), *respectively. Then we have* (1.24).

In Theorem 1.3, we have the same conclusion (1.24) even if the condition (1.23) is replaced with

$$\int_a^{\infty} q(s)F(s)^{\beta}\,ds < \infty \tag{1.27}$$

and

$$\text{there is a constant } C > 0 \quad \text{such that} \quad \int_t^{\infty} q(s)F(s)^{\beta}\,ds \leqslant C\int_t^{\infty} q(s)P(s)^{\beta}\,ds \quad \text{for all large } t, \tag{1.28}$$

where

$$F(t) = \left(\int_t^{\infty} q(r)\,dr\right)^{-1/\alpha}\int_a^{t}\left[\frac{1}{p(r)}\int_r^{\infty} q(\tau)\,d\tau\right]^{1/\alpha} dr.$$

Similarly, in Theorem 1.4, the conditions (1.25) and (1.26) can be replaced with

$$\text{there is a constant } C > 0 \quad \text{such that} \quad \int_a^{t} q(s)\Phi(s)^{\beta}\,ds \leqslant C\int_a^{t} q(s)\pi(s)^{\beta}\,ds \quad \text{for all large } t, \tag{1.29}$$

where

$$\Phi(t) = \left(\int_a^{t} q(r)\,dr\right)^{-1/\alpha}\int_t^{\infty}\left[\frac{1}{p(r)}\int_a^{r} q(\tau)\,d\tau\right]^{1/\alpha} dr.$$

Thus we can formulate these results as follows:

Theorem 1.5. *In Eqs.* (1.1) *and* (1.21), *let* $\alpha > \beta$ *and suppose that* (1.5), (1.22), (1.27) *and* (1.28) *hold. Let* $x(t)$ *and* $x_0(t)$ *be slowly growing positive solutions of* (1.1) *and* (1.21), *respectively. Then we have* (1.24).

Theorem 1.6. *In Eqs.* (1.1) *and* (1.21), *let* $\alpha > \beta$ *and suppose that* (1.6), (1.22) *and* (1.29) *hold. Let* $x(t)$ *and* $x_0(t)$ *be slowly decaying positive solutions of* (1.1) *and* (1.21), *respectively. Then we have* (1.24).

By Theorems 1.3 and 1.5 [*resp.* Theorems 1.4 and 1.6] applied to the special case $p_0(t) \equiv p(t)$ and $q_0(t) \equiv q(t)$, we have the following. Let $\varphi(t)$ be a fixed slowly growing [*resp.* slowly decaying] positive solution of (1.1). Then, for every slowly growing [*resp.* slowly decaying] positive solution $x(t)$ of (1.1), we have $x(t) \sim \varphi(t)$ as $t \to \infty$. In other words, if the condition (1.23) or (1.27)–(1.28) [*resp.* (1.25)–(1.26) or (1.29)] is satisfied, then any of slowly growing [*resp.* slowly decaying] positive solutions of (1.1) has the same asymptotic behavior as $t \to \infty$.

Theorem 1.4 has been proved by Kamo and Usami [4] under the restricted assumption $p_0 \equiv p \in C^1[a, \infty)$.

We can utilize Theorems 1.3 and 1.5 [*resp.* Theorems 1.4 and 1.6] for obtaining precise asymptotic forms of slowly growing [*resp.* slowly decaying] positive solutions of (1.1). For example, the following proposition can be derived.

Proposition 1.1. *Let $\alpha > \beta$ and let* (1.5) *be satisfied. Assume that*

$$q(t) \sim \frac{\kappa}{p(t)^{1/\alpha}} P(t)^{-\mu}\big(\log P(t)\big)^{\rho} \quad \text{as } t \to \infty,$$

where κ, μ and ρ are constants such that $\kappa > 0$, $\beta + 1 < \mu < \alpha + 1$ and $\rho \in \boldsymbol{R}$. Then, (1.1) *has a slowly growing positive solution. Moreover, for any slowly growing positive solution $x(t)$ of* (1.1), *we have*

$$x(t) \sim \left\{\frac{\kappa}{\alpha(1-\nu)\nu^{\alpha}}\right\}^{1/(\alpha-\beta)} P(t)^{\nu}\big(\log P(t)\big)^{\rho/(\alpha-\beta)} \quad \text{as } t \to \infty,$$

where $\nu = (\alpha - \mu + 1)/(\alpha - \beta)$.

Proposition 1.1 is a generalization of (i) of Theorem 1.1 in [3]. A generalization of (ii) of Theorem 1.1 in [3] can be similarly obtained. More general results including Proposition 1.1 are given in Section 4.

In Section 2 of this paper, we give preparatory results concerning slowly growing positive solutions of (1.1). Theorem 1.2 is proved also in Section 2. The proofs of Theorems 1.3–1.6 are given in Section 3. Finally, in Section 4, we state and prove general results on precise asymptotic forms of slowly growing and slowly decaying positive solutions of (1.1).

2. Preparatory results and the proof of Theorem 1.2

In this section, after showing preparatory results, we prove Theorem 1.2. Throughout this section the integral condition (1.5) is assumed to hold.

Let $x(t)$ be an eventually positive solution of (1.1). There is $T \geqslant a$ such that $x(t) > 0$ for $t \geqslant T$. By (1.1),

$$\big(p(t)\big|x'(t)\big|^{\alpha} \operatorname{sgn} x'(t)\big)' = -q(t)\big|x(t)\big|^{\beta} \operatorname{sgn} x(t) \leqslant 0$$

for $t \geqslant T$, and so $p(t)|x'(t)|^{\alpha} \operatorname{sgn} x'(t)$ is decreasing on $[T, \infty)$. The function $p(t)|x'(t)|^{\alpha} \operatorname{sgn} x'(t)$ is nonnegative for $t \geqslant T$. Indeed, if there is $T_1 \geqslant T$ such that $p(T_1)|x'(T_1)|^{\alpha} \operatorname{sgn} x'(T_1) < 0$, then

$$p(t)\big|x'(t)\big|^{\alpha} \operatorname{sgn} x'(t) \leqslant p(T_1)\big|x'(T_1)\big|^{\alpha} \operatorname{sgn} x'(T_1) \equiv -c_0 < 0 \quad (c_0 > 0)$$

for $t \geqslant T_1$. This implies $x'(t) < 0$ for $t \geqslant T_1$ and

$$x'(t) \leqslant -\frac{c_0^{1/\alpha}}{p(t)^{1/\alpha}}, \quad t \geqslant T_1.$$

Integrating the above inequality from T_1 to t, we have

$$x(t) \leqslant x(T_1) - c_0^{1/\alpha} \int_{T_1}^{t} \frac{1}{p(s)^{1/\alpha}}\, ds, \quad t \geqslant T_1.$$

Then, (1.5) implies $x(t) \to -\infty$ $(t \to \infty)$, which contradicts the condition that $x(t) > 0$ for $t \geqslant T$. Thus, $p(t)|x'(t)|^{\alpha} \times \operatorname{sgn} x'(t) \geqslant 0$ for $t \geqslant T$. If there is $T_1 \geqslant T$ such that $p(T_1)|x'(T_1)|^{\alpha} \operatorname{sgn} x'(T_1) = 0$, then, since $p(t)|x'(t)|^{\alpha} \operatorname{sgn} x'(t)$ is decreasing and nonnegative on $[T, \infty)$, we get

$$p(t)\big|x'(t)\big|^{\alpha} \operatorname{sgn} x'(t) \equiv 0$$

on $[T_1, \infty)$. Then, by (1.1), $q(t) \equiv 0$ on $[T_1, \infty)$. This is a contradiction to the hypothesis that $q(t) \not\equiv 0$ on $[b, \infty)$ for any $b \geqslant a$. Consequently we find that

$$p(t)\big|x'(t)\big|^{\alpha} \operatorname{sgn} x'(t) > 0, \quad t \geqslant T.$$

In particular, $x'(t) > 0$ for $t \geqslant T$, and so (1.1) is reduced to

$$\big(p(t)x'(t)^{\alpha}\big)' + q(t)x(t)^{\beta} = 0 \tag{2.1}$$

for $t \geqslant T$. Since $p(t)x'(t)^{\alpha}$ is decreasing and positive on $[T, \infty)$, the limit $\lim_{t\to\infty} p(t)x'(t)^{\alpha} \equiv l$ exists and is a nonnegative value. It is easily seen that

$$\lim_{t\to\infty} \frac{x(t)}{P(t)} = \lim_{t\to\infty} p(t)^{1/\alpha} x'(t) = l^{1/\alpha}.$$

Therefore, if $x(t)$ is a slowly growing positive solution of (1.1), then $l = 0$.

In what follows we suppose that $x(t)$ is a slowly growing positive solution of (1.1). As stated just now, $\lim_{t\to\infty} p(t)x'(t)^\alpha = 0$. Integrate (2.1) from t ($\geqslant T$) to τ, and let $\tau \to \infty$. Then

$$p(t)x'(t)^\alpha = \int_t^\infty q(s)x(s)^\beta\,ds \tag{2.2}$$

for $t \geqslant T$. Since $x'(t) > 0$ for $t \geqslant T$, $x(t)$ is (strictly) increasing on $[T,\infty)$. Hence, (2.2) shows that

$$p(t)x'(t)^\alpha \geqslant x(t)^\beta \int_t^\infty q(s)\,ds, \quad t \geqslant T. \tag{2.3}$$

To shorten notation, we set

$$Q(t) \equiv \int_t^\infty q(s)\,ds.$$

The function $p(t)x'(t)^\alpha/Q(t)$ is increasing on $[T,\infty)$ since

$$\frac{d}{dt}\left[\frac{p(t)x'(t)^\alpha}{Q(t)}\right] = \frac{q(t)[p(t)x'(t)^\alpha - x(t)^\beta Q(t)]}{Q(t)^2} \geqslant 0$$

for $t \geqslant T$, where (2.1) and (2.3) have been used.

If $\alpha > \beta$, then

$$\liminf_{t\to\infty} \frac{p(t)x'(t)^\alpha}{Q(t)x(t)^\alpha} = 0. \tag{2.4}$$

In fact, assume to the contrary that (2.4) does not hold. There exist constants $c > 0$ and $T_1 \geqslant T$ such that

$$p(t)x'(t)^\alpha \geqslant c\,Q(t)x(t)^\alpha$$

for $t \geqslant T_1$. By (2.1) and the condition $\alpha > \beta$, the above inequality is rewritten as

$$\frac{d}{dt}\left[p(t)x'(t)^\alpha\right]^{(\alpha-\beta)/\alpha} \geqslant c^{-\beta/\alpha}\frac{d}{dt}Q(t)^{(\alpha-\beta)/\alpha}$$

for $t \geqslant T_1$. Then, an integration of this inequality over $[t,\infty)$ gives

$$p(t)x'(t)^\alpha \leqslant c^{-\beta/(\alpha-\beta)}Q(t), \quad t \geqslant T_1,$$

and it follows from (2.3) that $x(t) \leqslant c^{-1/(\alpha-\beta)}$ for $t \geqslant T_1$. This is a contradiction to the condition that $x(t) \to \infty$ as $t \to \infty$. Thus, if $\alpha > \beta$, then we have (2.4).

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. As mentioned in Section 1, the "if" part has been proved by Elbert and Kusano [2]. We will prove the "only if" part. Let $\alpha > \beta$ and suppose that (1.1) has a slowly growing positive solution $x(t)$. As stated in Section 1, it is well known that, for the case $\alpha > \beta$, Eq. (1.1) has an eventually positive solution (with no conditions such as (1.8)–(1.10)) if and only if the first integral condition of (1.20) holds. Thus we have only to prove the second integral condition of (1.20).

Let $x(t) > 0$ for $t \geqslant T$. Because of $x(t) \to \infty$ ($t \to \infty$), there is a sufficiently large T_1 ($\geqslant T$) such that

$$\frac{1}{2}x(t) \leqslant x(t) - x(T) = \int_T^t x'(s)\,ds \tag{2.5}$$

for $t \geqslant T_1$. Note the equality

$$\int_T^t x'(s)\,ds = \int_T^t \left[\frac{Q(s)}{p(s)}\right]^{1/\alpha}\left[\frac{p(s)x'(s)^\alpha}{Q(s)}\right]^{1/\alpha} ds \tag{2.6}$$

and recall the fact that $p(t)x'(t)^\alpha/Q(t)$ is increasing on $[T,\infty)$. Then it follows from (2.5) and (2.6) that

$$\frac{1}{2} \leqslant \left[\frac{p(t)x'(t)^\alpha}{x(t)^\alpha Q(t)}\right]^{1/\alpha}\int_T^t \left[\frac{Q(s)}{p(s)}\right]^{1/\alpha} ds \tag{2.7}$$

for $t \geqslant T_1$. Taking the lim inf as $t \to \infty$ in (2.7) and using (2.4), we conclude that the second integral condition of (1.20) holds. The proof of Theorem 1.2 is complete. □

3. Proofs of Theorems 1.3–1.6

In this section we give the proofs of Theorems 1.3–1.6. To prove Theorem 1.3, we first obtain asymptotic estimates of slowly growing positive solutions of (1.1). Then the results in Section 2 are used effectively.

Lemma 3.1. *Suppose that* (1.5) *holds. Let $\alpha > \beta$. Let $x(t)$ be a slowly growing positive solution of* (1.1), *and let $x(t) > 0$ for $t \geqslant T$. Then*

$$x(t) \geqslant \left[x(T)^{(\alpha-\beta)/\alpha} + \frac{\alpha-\beta}{\alpha}\int_T^t \left[\frac{1}{p(s)}\int_s^\infty q(r)\,dr\right]^{1/\alpha} ds\right]^{\alpha/(\alpha-\beta)} \tag{3.1}$$

for $t \geqslant T$, and

$$x(t) \leqslant \left[x(T_1)^{(\alpha-\beta)/\alpha} + \frac{\alpha-\beta}{\alpha}\int_{T_1}^t \left[\frac{1}{p(s)P(s)^\beta}\int_s^\infty q(r)P(r)^\beta\,dr\right]^{1/\alpha} ds\right]^{\alpha/(\alpha-\beta)} \tag{3.2}$$

for $t \geqslant T_1$, where T_1 $(\geqslant T)$ is taken sufficiently large.

Proof. As shown in Section 2, we have (2.3), or equivalently

$$\frac{x'(t)}{x(t)^{\beta/\alpha}} \geqslant \left[\frac{1}{p(t)}\int_t^\infty q(s)\,ds\right]^{1/\alpha}, \quad t \geqslant T. \tag{3.3}$$

An integration of (3.3) over $[T,t]$ yields (3.1).

The function $p(t)x'(t)^\alpha$ is decreasing and positive on $[T,\infty)$, and so is the function $p(t)^{1/\alpha}x'(t)$. Hence

$$x(t) = x(T) + \int_T^t \frac{p(s)^{1/\alpha}x'(s)}{p(s)^{1/\alpha}}\,ds \geqslant x(T) + p(t)^{1/\alpha}x'(t)\big[P(t) - P(T)\big], \quad t \geqslant T,$$

where $P(t)$ is defined by (1.7). Since $x(T) - p(t)^{1/\alpha}x'(t)P(T) \to x(T)$ (> 0) as $t \to \infty$, if T_1 is sufficiently large, then

$$x(t) > p(t)^{1/\alpha}x'(t)P(t), \quad t \geqslant T_1.$$

This means that $[x(t)/P(t)]' < 0$ $(t \geqslant T_1)$, and so $x(t)/P(t)$ is (strictly) decreasing on $[T_1,\infty)$. Then it follows from (2.2) that

$$p(t)x'(t)^\alpha \leqslant \left[\frac{x(t)}{P(t)}\right]^\beta \int_t^\infty q(s)P(s)^\beta\,ds, \quad t \geqslant T_1.$$

Note that the infinite integral in the right-hand side of the above inequality is well defined (see Theorem 1.2). Rewrite the above inequality as

$$\frac{x'(t)}{x(t)^{\beta/\alpha}} \leqslant \left[\frac{1}{p(t)P(t)^\beta}\int_t^\infty q(s)P(s)^\beta\,ds\right]^{1/\alpha}, \quad t \geqslant T_1, \tag{3.4}$$

and integrate (3.4) over $[T_1,t]$. Then we get (3.2). The proof of Lemma 3.1 is complete. □

In Theorems 1.3–1.6, the auxiliary equation (1.21) as well as Eq. (1.1) are considered. Then, similarly to (1.7) and (1.13), we set

$$P_0(t) = \int_a^t \frac{1}{p_0(s)^{1/\alpha}}\,ds \quad \text{for the case} \quad \int_a^\infty \frac{1}{p_0(s)^{1/\alpha}}\,ds = \infty, \tag{3.5}$$

and

$$\pi_0(t) = \int_t^\infty \frac{1}{p_0(s)^{1/\alpha}}\,ds \quad \text{for the case} \quad \int_a^\infty \frac{1}{p_0(s)^{1/\alpha}}\,ds < \infty. \tag{3.6}$$

We make use of the following lemma for the proofs of Theorem 1.3 and Theorem 1.4. The proof is left to the reader.

Lemma 3.2. *Let $f, g \in C^1[T,\infty)$. Let*

$$\lim_{t\to\infty} g(t) = \infty \quad \text{and} \quad g'(t) > 0 \quad \text{for all large } t. \tag{3.7}$$

Then

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \leqslant \liminf_{t\to\infty} \frac{f(t)}{g(t)} \leqslant \limsup_{t\to\infty} \frac{f(t)}{g(t)} \leqslant \limsup_{t\to\infty} \frac{f'(t)}{g'(t)}. \tag{3.8}$$

If we replace (3.7) *with the condition*

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0 \quad \text{and} \quad g'(t) < 0 \quad \text{for all large } t, \tag{3.9}$$

then the same conclusion (3.8) *holds.*

Proof of Theorem 1.3. Let $x(t)$ be a slowly growing positive solution of (1.1), and let $x_0(t)$ be a slowly growing positive solution of (1.21). Note that $x(t)$ satisfies (2.2), and likewise $x_0(t)$ satisfies (2.2) with x, p, q replaced by x_0, p_0, q_0. Further, by Lemma 3.1, $x(t)$ satisfies (3.1) and (3.2) for all large t, and $x_0(t)$ satisfies (3.1) and (3.2) with x, p, q, P replaced by x_0, p_0, q_0, P_0 for all large t. Here, T and T_1 $(\geqslant T)$ are chosen sufficiently large. Then, on account of (1.22) and (1.23), we find that

$$0 < \liminf_{t\to\infty} \frac{x(t)}{x_0(t)} \leqslant \limsup_{t\to\infty} \frac{x(t)}{x_0(t)} < \infty. \tag{3.10}$$

Employing Lemma 3.2, we compute as follows:

$$\begin{aligned}
\Lambda \equiv \limsup_{t\to\infty} \frac{x(t)}{x_0(t)} &\leqslant \limsup_{t\to\infty} \frac{x'(t)}{x_0'(t)} \\
&= \limsup_{t\to\infty} \frac{[\frac{1}{p(t)} \int_t^\infty q(s)x(s)^\beta\, ds]^{1/\alpha}}{[\frac{1}{p_0(t)} \int_t^\infty q_0(s)x_0(s)^\beta\, ds]^{1/\alpha}} \\
&= \left[\limsup_{t\to\infty} \frac{\int_t^\infty q(s)x(s)^\beta\, ds}{\int_t^\infty q_0(s)x_0(s)^\beta\, ds}\right]^{1/\alpha} \\
&\leqslant \left[\limsup_{t\to\infty} \frac{-q(t)x(t)^\beta}{-q_0(t)x_0(t)^\beta}\right]^{1/\alpha} = \left[\limsup_{t\to\infty} \frac{x(t)}{x_0(t)}\right]^{\beta/\alpha} = \Lambda^{\beta/\alpha},
\end{aligned}$$

where (1.22) has been used. Since $0 < \beta/\alpha < 1$ and $0 < \Lambda < \infty$ (see (3.10)), the above fact implies

$$0 < \Lambda \leqslant 1. \tag{3.11}$$

In the same manner we can see that $\lambda \equiv \liminf_{t\to\infty}[x(t)/x_0(t)]$ satisfies

$$1 \leqslant \lambda < \infty. \tag{3.12}$$

Then, by (3.11), (3.12) and the trivial inequality $\lambda \leqslant \Lambda$, we obtain $\lambda = \Lambda = 1$. This means (1.24). The proof of Theorem 1.3 is complete. □

The following lemma, which corresponds to Lemma 3.1, has been essentially shown in [4, Lemma 3.1].

Lemma 3.3. *Suppose that* (1.6) *and* (1.25) *hold. Let $\alpha > \beta$. Suppose that $x(t)$ is a slowly decaying positive solution of* (1.1)*, and let $x(t) > 0$ for $t \geqslant T$. Then*

$$\left[\frac{\alpha-\beta}{\alpha}\int_t^\infty\left[\frac{1}{p(s)}\int_{T_1}^s q(r)\,dr\right]^{1/\alpha} ds\right]^{\alpha/(\alpha-\beta)} \leqslant x(t) \leqslant \left[2\frac{\alpha-\beta}{\alpha}\int_t^\infty\left[\frac{1}{p(s)\pi(s)^\beta}\int_{T_1}^s q(r)\pi(r)^\beta\,dr\right]^{1/\alpha} ds\right]^{\alpha/(\alpha-\beta)} \tag{3.13}$$

for $t \geqslant T_1$, where T_1 $(\geqslant T)$ is taken sufficiently large.

Proof of Theorem 1.4. Let $x(t)$ and $x_0(t)$ be slowly decaying positive solutions of (1.1) and (1.21), respectively. Arguments similar to that in the proof of Theorem 1.3 show that $\Lambda \equiv \limsup_{t\to\infty} \frac{x(t)}{x_0(t)}$ and $\lambda \equiv \liminf_{t\to\infty} \frac{x(t)}{x_0(t)}$ satisfy

$$0 < \lambda \leqslant \Lambda < \infty, \qquad 0 < \Lambda \leqslant 1 \quad \text{and} \quad 1 \leqslant \lambda < \infty. \tag{3.14}$$

The detailed verification of (3.14) is left to the reader. The fact (3.14) implies $\lambda = \Lambda = 1$, and hence we have (1.24). The proof of Theorem 1.4 is complete. □

Proof of Theorem 1.5. Suppose that $x(t)$ is a slowly growing positive solution of (1.1). Let $x(t)>0$ for $t \geqslant T$. As in Section 2, we see that $x'(t)>0$ for $t \geqslant T$ and that $\int_T^\infty q(s)\,ds$ exists. Since $q(t) \sim q_0(t)$ $(t \to \infty)$ and $q_0(t)>0$ for $t \geqslant a$, we may suppose $q(t)>0$ for $t \geqslant T$.

An easy calculation shows that

$$y(t) \equiv p(t)\left|x'(t)\right|^{\alpha} \operatorname{sgn} x'(t)$$

is a positive solution of the equation

$$\left(p^*(t)\left|y'\right|^{\alpha^*} \operatorname{sgn} y'\right)' + q^*(t)|y|^{\beta^*} \operatorname{sgn} y = 0, \quad t \geqslant T, \tag{3.15}$$

where

$$p^*(t) = q(t)^{-1/\beta}, \qquad q^*(t) = p(t)^{-1/\alpha}, \qquad \alpha^* = \frac{1}{\beta} \quad \text{and} \quad \beta^* = \frac{1}{\alpha}.$$

We have

$$\int_T^\infty \frac{1}{p^*(t)^{1/\alpha^*}}\,dt = \int_T^\infty q(t)\,dt < \infty \quad \text{and}$$

$$\pi^*(t) \equiv \int_t^\infty \frac{1}{p^*(s)^{1/\alpha^*}}\,ds = \int_t^\infty q(s)\,ds.$$

It is easy to see that

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(p(t)\left|x'(t)\right|^{\alpha} \operatorname{sgn} x'(t)\right) = \lim_{t\to\infty}\left(\frac{x(t)}{P(t)}\right)^{\alpha} = 0 \quad \text{and}$$

$$\lim_{t\to\infty} \frac{y(t)}{\pi^*(t)} = \lim_{t\to\infty}\left(-\frac{y'(t)}{q(t)}\right) = \lim_{t\to\infty} x(t)^{\beta} = \infty.$$

Thus $y=y(t)$ is a slowly decaying positive solution of (3.15).

In the same manner, if $x_0(t)$ is a slowly growing positive solution of (1.21) such that $x_0(t)>0$ $(t \geqslant T)$, then the function

$$y_0(t) \equiv p_0(t)\left|x_0'(t)\right|^{\alpha} \operatorname{sgn} x_0'(t)$$

is a slowly decaying positive solution of

$$\left(p_0^*(t)\left|y'\right|^{\alpha^*} \operatorname{sgn} y'\right)' + q_0^*(t)|y|^{\beta^*} \operatorname{sgn} y = 0, \quad t \geqslant T, \tag{3.16}$$

where

$$p_0^*(t) = q_0(t)^{-1/\beta}, \qquad q_0^*(t) = p_0(t)^{-1/\alpha}, \qquad \alpha^* = \frac{1}{\beta} \quad \text{and} \quad \beta^* = \frac{1}{\alpha}.$$

Observe here that (1.27) and (1.28) imply (1.25) and (1.26) with α, β, $p(t), q(t)$ and $\pi(t)$ replaced by α^*, β^*, $p^*(t), q^*(t)$ and $\pi^*(t)$, respectively. Then we can apply Theorem 1.4 to Eqs. (3.15) and (3.16), and, in consequence,

$$y(t) \sim y_0(t) \quad (t \to \infty). \tag{3.17}$$

Thus we have

$$\lim_{t\to\infty} \frac{x(t)}{x_0(t)} = \lim_{t\to\infty} \frac{x'(t)}{x_0'(t)} = \lim_{t\to\infty}\left(\frac{p_0(t)}{p(t)}\right)^{1/\alpha}\left(\frac{y(t)}{y_0(t)}\right)^{1/\alpha} = 1.$$

The proof of Theorem 1.5 is complete. □

Proof of Theorem 1.6. Let $x(t)$ and $x_0(t)$ be slowly decaying positive solutions of (1.1) and (1.21), respectively. We use the same notation as in the proof of Theorem 1.5. Then we have

$$\int_T^\infty \frac{1}{p^*(t)^{1/\alpha^*}}\,dt = \int_T^\infty q(t)\,dt = \infty \quad \text{and}$$

$$P^*(t) \equiv \int_T^t \frac{1}{p^*(s)^{1/\alpha^*}}\,ds = \int_T^t q(s)\,ds.$$

We can show that

$$y(t) \equiv -p(t)\left|x'(t)\right|^{\alpha} \operatorname{sgn} x'(t) \quad \text{and} \quad y_0(t) \equiv -p_0(t)\left|x_0'(t)\right|^{\alpha} \operatorname{sgn} x_0'(t)$$

are slowly growing positive solutions of (3.15) and (3.16), respectively. The condition (1.29) implies the condition (1.23) with α, β, $p(t), q(t)$ and $P(t)$ replaced by α^*, β^*, $p^*(t), q^*(t)$ and $P^*(t)$, respectively. Apply Theorem 1.3 to Eqs. (3.15) and (3.16). Then we obtain (3.17). The rest of the proof runs as the proof of Theorem 1.5. □

4. Asymptotic forms of slowly growing and slowly decaying positive solutions

In this section we use Theorems 1.3–1.6 for obtaining precise asymptotic forms of slowly growing and slowly decaying positive solutions of (1.1). Before stating the main results of this section, we give an illustrative example. Consider the equation

$$\left(t^{\gamma}\left|x'\right|^{\alpha} \operatorname{sgn} x'\right)' + Kt^{-\delta}|x|^{\beta} \operatorname{sgn} x = 0, \quad t \geqslant 1, \tag{4.1}$$

where $\alpha > \beta > 0$, $K > 0$ and $\gamma, \delta \in \boldsymbol{R}$ are constants. Clearly, $p(t) \equiv t^{\gamma}$ satisfies (1.5) if and only if $\alpha \geqslant \gamma$. Then

$$P(t) \sim \begin{cases} \frac{\alpha}{\alpha-\gamma} t^{(\alpha-\gamma)/\alpha} & (\alpha > \gamma), \\ \log t & (\alpha = \gamma). \end{cases}$$

By Theorem 1.2, Eq. (4.1) has a slowly growing positive solution if and only if

$$\beta - \frac{\beta}{\alpha}\gamma + 1 < \delta \leqslant \alpha - \gamma + 1. \tag{4.2}$$

Thus, for the case $\alpha = \gamma$, Eq. (4.1) does not have a slowly growing positive solution. For the case $\alpha > \gamma$, Eq. (4.1) may be rewritten in the form

$$\left(t^{\gamma}\left|x'\right|^{\alpha} \operatorname{sgn} x'\right)' + \frac{\kappa}{t^{\gamma/\alpha}}\left(\frac{\alpha}{\alpha-\gamma} t^{(\alpha-\gamma)/\alpha}\right)^{-\mu} |x|^{\beta} \operatorname{sgn} x = 0, \quad t \geqslant 1, \tag{4.3}$$

where $\kappa = K\alpha^{\mu}(\alpha-\gamma)^{-\mu}$, $\mu = (\alpha\delta - \gamma)/(\alpha - \gamma)$. Then, (4.2) becomes $\beta + 1 < \mu \leqslant \alpha + 1$. If $\beta - (\beta/\alpha)\gamma + 1 < \delta < \alpha - \gamma + 1$, or equivalently $\beta + 1 < \mu < \alpha + 1$, then (4.1) has a slowly growing positive solution $x = x_0(t)$ of the exact form

$$x_0(t) = \left\{\frac{K}{\eta^{\alpha}(\alpha - \gamma - \alpha\eta)}\right\}^{1/(\alpha-\beta)} t^{\eta}, \tag{4.4}$$

where $\eta = (\alpha - \gamma + 1 - \delta)/(\alpha - \beta)$. Note that $x_0(t)$ may be represented as

$$x_0(t) = \left\{\frac{\kappa}{\alpha(1-\nu)\nu^{\alpha}}\right\}^{1/(\alpha-\beta)} \left(\frac{\alpha}{\alpha-\gamma} t^{(\alpha-\gamma)/\alpha}\right)^{\nu}, \tag{4.5}$$

where $\nu = (\alpha - \mu + 1)/(\alpha - \beta)$. Therefore, by Theorem 1.3, we can conclude that if $q(t) \sim Kt^{-\delta}$ as $t \to \infty$, then every slowly growing positive solution $x(t)$ of

$$\left(t^{\gamma}\left|x'\right|^{\alpha} \operatorname{sgn} x'\right)' + q(t)|x|^{\beta} \operatorname{sgn} x = 0, \quad t \geqslant 1,$$

satisfies $x(t) \sim x_0(t)$ as $t \to \infty$, where $x_0(t)$ is given by (4.4) or (4.5).

The above fact can be formulated as a more general result in the following way.

Theorem 4.1. *In Eq.* (1.1), *let $\alpha > \beta$ and let* (1.5) *be satisfied. Let*

$$p(t) \sim p_0(t) \quad \text{and} \quad q(t) \sim \frac{\kappa}{p_0(t)^{1/\alpha}} P_0(t)^{-\mu} \omega_0\big(P_0(t)\big) \quad (t \to \infty), \tag{4.6}$$

where κ and μ are constants such that $\kappa > 0$ and $\beta + 1 < \mu < \alpha + 1$, and $P_0(t)$ is defined by (3.5), *and $\omega_0(\tau) > 0$ is a continuously differentiable function on an interval $[\tau_0, \infty)$, $\tau_0 > 1$, such that*

$$\lim_{\tau \to \infty} \frac{\tau \dot{\omega}_0(\tau)}{\omega_0(\tau)} = 0 \quad \left(\dot{} = \frac{d}{d\tau}\right). \tag{4.7}$$

Then, (1.1) *has a slowly growing positive solution. Moreover, every slowly growing positive solution $x(t)$ of* (1.1) *satisfies*

$$x(t) \sim \left\{\frac{\kappa}{\alpha(1-\nu)\nu^{\alpha}}\right\}^{1/(\alpha-\beta)} P_0(t)^{\nu} \omega_0\big(P_0(t)\big)^{1/(\alpha-\beta)} \quad (t \to \infty),$$

where $\nu = (\alpha - \mu + 1)/(\alpha - \beta)$.

Proof. Define the function $\widehat{q}_0(t)$ on $[t_0, \infty)$ by

$$\widehat{q}_0(t) = \frac{\kappa}{p_0(t)^{1/\alpha}} P_0(t)^{-\mu}\omega_0\big(P_0(t)\big), \quad t \geqslant t_0.$$

Here, t_0 is chosen so that $P_0(t) \geqslant \tau_0$ for $t \geqslant t_0$.

As a general result, a continuously differentiable function $\omega_0(\tau) > 0$ which satisfies (4.7) has the following properties.

(i) For any $\varepsilon > 0$, we have

$$\limsup_{\tau\to\infty} \tau^{-\varepsilon}\omega_0(\tau) < \infty \quad \text{and} \quad \liminf_{\tau\to\infty} \tau^{\varepsilon}\omega_0(\tau) > 0.$$

(ii) Let $\gamma > -1$ and $\delta \in \boldsymbol{R}$. Then

$$\int_{\tau_0}^{\infty} u^{\gamma}\omega_0(u)^{\delta}\,du = \infty \quad \text{and}$$

$$\int_{\tau_0}^{\tau} u^{\gamma}\omega_0(u)^{\delta}\,du \sim \frac{1}{\gamma+1}\tau^{\gamma+1}\omega_0(\tau)^{\delta} \quad (\tau \to \infty).$$

(iii) Let $\gamma < -1$ and $\delta \in \boldsymbol{R}$. Then

$$\int_{\tau_0}^{\infty} u^{\gamma}\omega_0(u)^{\delta}\,du < \infty \quad \text{and}$$

$$\int_{\tau}^{\infty} u^{\gamma}\omega_0(u)^{\delta}\,du \sim -\frac{1}{\gamma+1}\tau^{\gamma+1}\omega_0(\tau)^{\delta} \quad (\tau \to \infty).$$

Therefore, by using these facts (i)–(iii), we can prove (1.23) with $p(t), q(t)$ and $P(t)$ replaced by $p_0(t), \widehat{q}_0(t)$ and $P_0(t)$, respectively. The detailed verification is left to the reader.

Now, define the function $y_0(t)$ by

$$y_0(t) = \left\{\frac{\kappa}{\alpha(1-\nu)\nu^{\alpha}}\right\}^{1/(\alpha-\beta)} P_0(t)^{\nu}\omega_0\big(P_0(t)\big)^{1/(\alpha-\beta)}, \quad t \geqslant t_0.$$

Using the above facts (i)–(iii), we find that

$$\int_{t_0}^{\infty} \widehat{q}_0(r)y_0(r)^{\beta}\,dr < \infty \quad \text{and}$$

$$\int_{s}^{\infty} \widehat{q}_0(r)y_0(r)^{\beta}\,dr \sim \kappa\left\{\frac{\kappa}{\alpha(1-\nu)\nu^{\alpha}}\right\}^{\beta/(\alpha-\beta)} \frac{\alpha-\beta}{\alpha(\mu-\beta-1)} P_0(s)^{-\alpha(\mu-\beta-1)/(\alpha-\beta)}\omega_0\big(P_0(s)\big)^{\alpha/(\alpha-\beta)} \quad (s \to \infty).$$

Therefore it is possible to define $x_0(t)$ by

$$x_0(t) = \int_{t_0}^{t} \frac{1}{p_0(s)^{1/\alpha}} \left(\int_{s}^{\infty} \widehat{q}_0(r)y_0(r)^{\beta}\,dr\right)^{1/\alpha} ds, \quad t \geqslant t_0.$$

Then, $x_0(t)$ satisfies

$$x_0(t) \sim \left\{\frac{\kappa}{\alpha(1-\nu)\nu^{\alpha}}\right\}^{1/(\alpha-\beta)} P_0(t)^{\nu}\omega_0\big(P_0(t)\big)^{1/(\alpha-\beta)} \quad (t \to \infty),$$

and so

$$x_0(t) \sim y_0(t) \quad (t \to \infty).$$

Furthermore

$$\lim_{t\to\infty} x_0(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{x_0(t)}{P_0(t)} = 0.$$

It is easy to see that $x_0(t)$ satisfies

$$\big(p_0(t)x_0'(t)^{\alpha}\big)' + \widehat{q}_0(t)y_0(t)^{\beta} = 0,$$

hence, $x = x_0(t)$ is a slowly growing positive solution of (1.21) with

$$q_0(t) = \widehat{q}_0(t)y_0(t)^{\beta}x_0(t)^{-\beta}.$$

Since $x_0(t) \sim y_0(t)$ $(t \to \infty)$ and since (4.6) and (4.7) are assumed to hold, we have (1.22) as well as (1.23). Then, by Theorem 1.3, every slowly growing positive solution $x(t)$ of (1.1) satisfies $x(t) \sim x_0(t)$ $(t \to \infty)$. Using the property $x_0(t) \sim y_0(t)$ $(t \to \infty)$ again, we conclude $x(t) \sim y_0(t)$ $(t \to \infty)$. This completes the proof of Theorem 4.1. □

We may also apply Theorem 1.5 for the proof of Theorem 4.1, while Theorem 1.3 has been used in the above proof of Theorem 4.1.

A typical example of the function $\omega_0(\tau) > 0$ satisfying (4.7) is $\omega_0(\tau) = (\log \tau)^{\rho}$, $\rho \in \boldsymbol{R}$, for which Theorem 4.1 yields Proposition 1.1.

Now, let us turn to the case $\alpha > \gamma$ and $\delta = \alpha - \gamma + 1$ (or $\mu = \alpha + 1$) in the previous equation (4.1) (or (4.3)). In this case, it seems difficult for us to get an exact solution of (4.1). But we can establish the following Theorem 4.2.

Theorem 4.2. *In Eq.* (1.1), *let* $\alpha > \beta$ *and let* (1.5) *be satisfied. Let*

$$p(t) \sim p_0(t) \quad \text{and} \quad q(t) \sim \frac{\kappa}{p_0(t)^{1/\alpha}} P_0(t)^{-\alpha-1}\omega_1\big(P_0(t)\big) \quad (t \to \infty),$$

where κ *is a positive constant, and* $P_0(t)$ *is defined by* (3.5), *and* $\omega_1(\tau) > 0$ *is a continuously differentiable function on an interval* $[\tau_0, \infty)$, $\tau_0 > 1$, *such that*

$$\lim_{\tau\to\infty} \frac{\tau(\log\tau)\dot{\omega}_1(\tau)}{\omega_1(\tau)} = 0 \quad \left(\cdot = \frac{d}{d\tau}\right). \tag{4.8}$$

Then, (1.1) *has a slowly growing positive solution. Moreover, every slowly growing positive solution* $x(t)$ *of* (1.1) *satisfies*

$$x(t) \sim \left\{\frac{\kappa(\alpha-\beta)^{\alpha}}{\alpha^{\alpha+1}}\right\}^{1/(\alpha-\beta)} \big(\log P_0(t)\big)^{\alpha/(\alpha-\beta)}\omega_1\big(P_0(t)\big)^{1/(\alpha-\beta)} \quad (t \to \infty).$$

The proof of Theorem 4.2 can be accomplished analogously to that of Theorem 4.1 by applying Theorem 1.3. Then we make use of the following facts. Let $\omega_1(\tau) > 0$ be a continuously differentiable function which satisfies (4.8). Then

(i) For any $\varepsilon > 0$, we have

$$\limsup_{\tau\to\infty} (\log\tau)^{-\varepsilon}\omega_1(\tau) < \infty \quad \text{and} \quad \liminf_{\tau\to\infty} (\log\tau)^{\varepsilon}\omega_1(\tau) > 0.$$

(ii) Let $\gamma > -1$ and $\delta \in \boldsymbol{R}$. Then

$$\int_{\tau_0}^{\infty} u^{-1}(\log u)^{\gamma}\omega_1(u)^{\delta}\,du = \infty \quad \text{and}$$

$$\int_{\tau_0}^{\tau} u^{-1}(\log u)^{\gamma}\omega_1(u)^{\delta}\,du \sim \frac{1}{\gamma+1}(\log\tau)^{\gamma+1}\omega_1(\tau)^{\delta} \quad (\tau \to \infty).$$

(iii) Let $\gamma < -1$, $\delta_1 \in \boldsymbol{R}$ and $\delta_2 \in \boldsymbol{R}$. Then

$$\int_{\tau_0}^{\infty} u^{\gamma}(\log u)^{\delta_1}\omega_1(u)^{\delta_2}\,du < \infty \quad \text{and}$$

$$\int_{\tau}^{\infty} u^{\gamma}(\log u)^{\delta_1}\omega_1(u)^{\delta_2}\,du \sim -\frac{1}{\gamma+1}\tau^{\gamma+1}(\log\tau)^{\delta_1}\omega_1(\tau)^{\delta_2} \quad (\tau \to \infty).$$

The detailed verification of Theorem 4.2 is left to the reader. It should be noticed here that, since (1.28) does not hold, Theorem 1.5 is unavailable for the proof of Theorem 4.2.

An example of the function $\omega_1(\tau) > 0$ which satisfies (4.8) is $\omega_1(\tau) = [\log(\log\tau)]^{\rho}$, $\rho \in \boldsymbol{R}$.

We can establish the following Theorems 4.3 and 4.4, which deal with the case (1.6). The proofs are also left to the reader. We remark that both of Theorems 1.4 and 1.6 are available for the proof of Theorem 4.3, and that Theorem 1.6 must be used for the proof of Theorem 4.4 since Theorems 1.4 is unavailable for the proof of Theorem 4.4.

Theorem 4.3. *In Eq.* (1.1), *let* $\alpha > \beta$ *and let* (1.6) *be satisfied. Let*

$$p(t) \sim p_0(t) \quad \text{and} \quad q(t) \sim \frac{\kappa}{p_0(t)^{1/\alpha}} \pi_0(t)^{-\mu} \omega_0\big(\pi_0(t)\big) \quad (t \to \infty),$$

where κ *and* μ *are constants such that* $\kappa > 0$ *and* $\beta + 1 < \mu < \alpha + 1$, *and* $\pi_0(t)$ *is defined by* (3.6), *and* $\omega_0(\tau) > 0$ *is a continuously differentiable function on an interval* $(0, \tau_0]$, $0 < \tau_0 < 1$, *such that*

$$\lim_{\tau \to +0} \frac{\tau \dot{\omega}_0(\tau)}{\omega_0(\tau)} = 0 \quad \left(\cdot = \frac{d}{d\tau} \right).$$

Then, (1.1) *has a slowly decaying positive solution. Moreover, every slowly decaying positive solution* $x(t)$ *of* (1.1) *satisfies*

$$x(t) \sim \left\{ \frac{\kappa}{\alpha(1-\nu)\nu^{\alpha}} \right\}^{1/(\alpha-\beta)} \pi_0(t)^{\nu} \omega_0\big(\pi_0(t)\big)^{1/(\alpha-\beta)} \quad (t \to \infty),$$

where $\nu = (\alpha - \mu + 1)/(\alpha - \beta)$.

Theorem 4.4. *In Eq.* (1.1), *let* $\alpha > \beta$ *and let* (1.6) *be satisfied. Let*

$$p(t) \sim p_0(t) \quad \text{and} \quad q(t) \sim \frac{\kappa}{p_0(t)^{1/\alpha}} \pi_0(t)^{-\beta-1} \omega_1\big(\pi_0(t)\big) \quad (t \to \infty),$$

where κ *is a positive constant, and* $\pi_0(t)$ *is defined by* (3.6), *and* $\omega_1(\tau) > 0$ *is a continuously differentiable function on an interval* $(0, \tau_0]$, $0 < \tau_0 < 1$, *such that*

$$\lim_{\tau \to +0} \frac{\tau |\log \tau| \dot{\omega}_1(\tau)}{\omega_1(\tau)} = 0 \quad \left(\cdot = \frac{d}{d\tau} \right).$$

Then, (1.1) *has a slowly decaying positive solution. Moreover, every slowly decaying positive solution* $x(t)$ *of* (1.1) *satisfies*

$$x(t) \sim \left\{ \frac{\kappa(\alpha-\beta)}{\alpha} \right\}^{1/(\alpha-\beta)} \pi_0(t) \big|\log \pi_0(t)\big|^{1/(\alpha-\beta)} \omega_1\big(\pi_0(t)\big)^{1/(\alpha-\beta)} \quad (t \to \infty).$$

Theorems 4.3 and 4.4 generalize the results of Kamo and Usami [4, Theorems 1.3 and 1.4].

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